Ćwiczenie nr 1: Stan przemieszczeń, odkształceń i naprężeń w punkcie.

Teoria sprężystości i plastyczności

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A. POLE DEFORMACJI:

Równania pola deformacji:

$x_1 = 1X_1 + 0X_2 + 3X_3$
$x_2 = -2X_1 + 4X_2 + 5X_3$ [x10^{-4} m]
$x_3 = 3X_1 + 4X_2 + 2X_3$

1. Funkcja składowych tensora odkształceń skończonych Lagrange'a:

$$L_{kl} = \frac{1}{2}\left(\frac{\delta x_i}{\delta X_k} \cdot \frac{\delta x_i}{\delta X_l} - \delta_{kl}\right)$$

$[L] = \frac{1}{2} \cdot ([G] \cdot [I])$ zapis tensorowy funkcji

1.1. Tensor deformacji Greena:

$$G_{kl} = \frac{\delta x_i}{\delta X_k} \cdot \frac{\delta x_i}{\delta X_l}$$

$[G] = \left[x_{i,j}\right]^T \left[x_{i,j}\right]$ postać macierzowa tensora

$\left[x_{i,j}\right]$ –macierz materialnych gradientów deformacji

$$\left[x_{i,j}\right] = \begin{bmatrix} x_{1.1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

$$x_{1,1} = \frac{\delta x_1}{\delta X_1} = 1$$
$$x_{1,2} = \frac{\delta x_1}{\delta X_2} = 0$$
$$x_{1,3} = \frac{\delta x_1}{\delta X_3} = 3$$

$$x_{2,1} = \frac{\delta x_2}{\delta X_1} = -2$$
$$x_{2,2} = \frac{\delta x_2}{\delta X_2} = 4$$
$$x_{2,3} = \frac{\delta x_2}{\delta X_3} = 5$$

$$x_{3,1} = \frac{\delta x_3}{\delta X_1} = 3$$
$$x_{3,2} = \frac{\delta x_3}{\delta X_2} = 4$$
$$x_{3,3} = \frac{\delta x_3}{\delta X_3} = 2$$

$$\left[x_{i,j}\right] = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \\ 3 & 4 & 2 \end{bmatrix}$$

$$[G] = [x_{i,j}]^T [x_{i,j}]$$

$$[G] = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \\ 3 & 4 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 4 & -1 \\ 4 & 32 & 28 \\ -1 & 28 & 38 \end{bmatrix}$$

1.2. Tensor odkształceń skończonych Lagrange'a:

$$[L] = \frac{1}{2} \cdot \left(\begin{bmatrix} 14 & 4 & -1 \\ 4 & 32 & 28 \\ -1 & 28 & 38 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 6,5 & 2 & -0,5 \\ 2 & 15,5 & 14 \\ -0,5 & 14 & 18,5 \end{bmatrix}$$

1.3. Tensor odkształceń skończonych Lagrange'a w zapisie wskaźnikowym:

$$L_{11} = \frac{1}{2} \cdot (x_{i,1} \cdot x_{i,1} - \delta_{11})$$
$$L_{11} = \frac{1}{2} \cdot (1 \cdot 1 + (-2) \cdot (-2) + 3 \cdot 3 - 1) = 6,5$$
$$L_{22} = \frac{1}{2} \cdot (x_{i,2} \cdot x_{i,2} - \delta_{22})$$
$$L_{22} = \frac{1}{2} \cdot (0 \cdot 0 + 4 \cdot 4 + 4 \cdot 4 - 1) = 15,5$$
$$L_{33} = \frac{1}{2} \cdot (x_{i,3} \cdot x_{i,3} - \delta_{33})$$
$$L_{33} = \frac{1}{2} \cdot (3 \cdot 3 + 5 \cdot 5 + 2 \cdot 2 - 1) = 18,5$$
$$L_{12} = \frac{1}{2} \cdot (x_{i,1} \cdot x_{i,2} - \delta_{12})$$
$$L_{12} = \frac{1}{2} \cdot (1 \cdot 0 + (-2) \cdot 4 + 3 \cdot 4 - 0) = 2$$
$$L_{13} = \frac{1}{2} \cdot (x_{i,1} \cdot x_{i,3} - \delta_{13})$$
$$L_{13} = \frac{1}{2} \cdot (1 \cdot 3 + (-2) \cdot 5 + 3 \cdot 2 - 0) = -0,5$$
$$L_{23} = \frac{1}{2} \cdot (x_{i,2} \cdot x_{i,3} - \delta_{23})$$
$$L_{23} = \frac{1}{2} \cdot (0 \cdot 3 + 4 \cdot 5 + 4 \cdot 2 - 0) = 14$$

$$[L] = \begin{bmatrix} 6,5 & 2 & -0,5 \\ 2 & 15,5 & 14 \\ -0,5 & 14 & 18,5 \end{bmatrix}$$

2. Funkcje składowych tensora odkształceń skończonych Eulera (Almansiego):

$x_1 = 1X_1 + 0X_2 + 3X_3$
$x_2 = -2X_1 + 4X_2 + 5X_3$ [x10^{-4} m]
$x_3 = 3X_1 + 4X_2 + 2X_3$

$$x_1 = 1X_1 + 3X_3 \Rightarrow X_1 = x_1 - 3X_3$$
$$x_2 = -2(x_1 - 3X_3) + 4X_2 + 5X_3 \Rightarrow X_2 = -\frac{11}{4}X_3 + \frac{1}{2}x_1 + \frac{1}{4}x_2$$
$$x_3 = 3(x_1 - 3X_3) + 4\left(-\frac{11}{4}X_3 + \frac{1}{2}x_1 + \frac{1}{4}x_2\right) + 2X_3 \Rightarrow X_3 = \frac{5}{18}x_1 + \frac{1}{18}x_2 - \frac{1}{18}x_3$$

$$X_1 = x_1 - 3\left(\frac{5}{18}x_1 + \frac{1}{18}x_2 - \frac{1}{18}x_3\right) = -\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{6}x_3$$
$$X_2 = -\frac{11}{4}\left(\frac{5}{18}x_1 + \frac{1}{18}x_2 - \frac{1}{18}x_3\right) + \frac{1}{2}x_1 + \frac{1}{4}x_2 = -\frac{19}{72}x_1 + \frac{7}{72}x_2 + \frac{11}{72}x_3$$

$$X_1 = -\frac{1}{6}x_1 - \frac{1}{6}x_2 + \frac{1}{6}x_3$$
$$X_2 = -\frac{19}{72}x_1 + \frac{7}{72}x_2 + \frac{11}{72}x_3$$
$$X_3 = \frac{5}{18}x_1 + \frac{1}{18}x_2 - \frac{1}{18}x_3$$

$E_{kl} = \frac{1}{2} \cdot \delta_{kl} - \frac{\delta X_i}{\delta x_k} \cdot \frac{\delta X_i}{\delta x_l}$ tensor odkształceń skończonych Eulera

$[E] = \frac{1}{2} \cdot ([I] - [C])$ zapis tensorowy

2.1. Tensor deformacji Cauchy'ego:

$$C_{kl} = \frac{\delta X_i}{\delta x_k} \cdot \frac{\delta X_i}{\delta x_l}$$

$[C] = [X_{i,j}]^T [X_{i,j}]$ postać macierzowa tensora

$$[X_{i,j}] = \begin{bmatrix} X_{1.1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{bmatrix}$$

$$X_{1,1} = \frac{\delta X_1}{\delta x_1} = -\frac{1}{6}$$
$$X_{1,2} = \frac{\delta X_1}{\delta x_2} = -\frac{1}{6}$$
$$X_{1,3} = \frac{\delta X_1}{\delta x_3} = \frac{1}{6}$$

$$X_{2,1} = \frac{\delta X_2}{\delta x_1} = -\frac{19}{72}$$
$$X_{2,2} = \frac{\delta X_2}{\delta x_2} = \frac{7}{72}$$
$$X_{2,3} = \frac{\delta X_2}{\delta x_3} = \frac{11}{72}$$

$$X_{3,1} = \frac{\delta X_3}{\delta x_1} = \frac{5}{18}$$
$$X_{3,2} = \frac{\delta X_3}{\delta x_2} = \frac{1}{18}$$
$$X_{3,3} = \frac{\delta X_3}{\delta x_3} = -\frac{1}{18}$$

$$[X_{i,j}] = \begin{bmatrix} -\frac{1}{6} & -\frac{19}{72} & \frac{5}{18} \\ -\frac{1}{6} & \frac{7}{72} & \frac{1}{18} \\ \frac{1}{6} & \frac{11}{72} & -\frac{1}{18} \end{bmatrix}$$

$$[C] = \begin{bmatrix} -\frac{1}{6} & -\frac{19}{72} & \frac{5}{18} \\ -\frac{1}{6} & \frac{7}{72} & \frac{1}{18} \\ \frac{1}{6} & \frac{11}{72} & -\frac{1}{18} \end{bmatrix}^T \cdot \begin{bmatrix} -\frac{1}{6} & -\frac{19}{72} & \frac{5}{18} \\ -\frac{1}{6} & \frac{7}{72} & \frac{1}{18} \\ \frac{1}{6} & \frac{11}{72} & -\frac{1}{18} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{23}{432} & -\frac{7}{108} \\ \frac{23}{432} & \frac{59}{576} & -\frac{11}{144} \\ -\frac{7}{108} & -\frac{11}{144} & \frac{1}{12} \end{bmatrix}$$

2.2. Tensor odkształceń skończonych Eulera:

$$[E] = \frac{1}{2} \cdot \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{12} & \frac{23}{432} & -\frac{7}{108} \\ \frac{23}{432} & \frac{59}{576} & -\frac{11}{144} \\ -\frac{7}{108} & -\frac{11}{144} & \frac{1}{12} \end{bmatrix} \right) = \begin{bmatrix} \frac{11}{24} & -\frac{23}{864} & \frac{7}{216} \\ -\frac{23}{864} & \frac{517}{1152} & \frac{11}{288} \\ \frac{7}{216} & \frac{11}{288} & \frac{11}{24} \end{bmatrix}$$

B. POLE PRZEMIESZCZEŃ:

Pole przemieszczeń we współrzędnych materialnych:

$u_1 = 2X_1^2 + 8X_2^2 - 1X_3^2 + 3X_1X_2 + 4X_2X_3 + 5X_3X_1 - 2X_1 - 4X_2 + 7X_3 + 6$
$u_2 = 0X_1^2 + 2X_2^2 - 4X_3^2 - 3X_1X_2 - X_2X_3 + 5X_3X_1 + 7X_1 + 6X_2 + 2X_3 + 0$ [x10^{-4} m]
$u_3 = 2X_1^2 - X_2^2 + 3X_3^2 + 4X_1X_2 - 3X_2X_3 + 2X_3X_1 + 0X_1 + 3X_2 + 1X_3 - 5$

1. Funkcje elementów tensora małych odkształceń:

$x_i = X_l$

$$\varepsilon_{ii} = \frac{\delta u_i}{\delta X_i}$$
$$\varepsilon_{ij} = \frac{1}{2} \cdot \left(\frac{\delta u_i}{\delta X_j} + \frac{\delta u_j}{\delta X_i}\right)$$

$\varepsilon_{11} = u_{1,1} = \frac{\delta u_1}{\delta X_1} = 4X_1 + 3X_2 + 5X_3 - 2$ [x10^{-4} m]
$\varepsilon_{22} = u_{2,2} = \frac{\delta u_2}{\delta X_2} = -3X_1 + 4X_2 - X_3 + 6$ [x10^{-4} m]
$\varepsilon_{33} = u_{3,3} = \frac{\delta u_3}{\delta X_3} = 2X_1 - 3X_2 + 6X_3 + 1$ [x10^{-4} m]
$\varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = \frac{1}{2}\left(\frac{\delta u_1}{\delta X_2} + \frac{\delta u_2}{\delta X_1}\right) = \frac{1}{2}\big((16X_2 + 3X_1 + 4X_3 - 4) + (-3X_2 + 5X_3 + 7)\big) = 1{,}5X_1 + 6{,}5X_2 + 4{,}5X_3 + 1{,}5$ [x10^{-4} m]
$\varepsilon_{13} = \frac{1}{2}(u_{1,3} + u_{3,1}) = \frac{1}{2}\left(\frac{\delta u_1}{\delta X_3} + \frac{\delta u_3}{\delta X_1}\right) = \frac{1}{2}\big((-2X_3 + 4X_2 + 5X_1 + 7) + (4X_1 + 4X_2 + 2X_3)\big) = 4{,}5X_1 + 4X_2 + 3{,}5$ [x10^{-4} m]
$\varepsilon_{23} = \frac{1}{2}(u_{2,3} + u_{3,2}) = \frac{1}{2}\left(\frac{\delta u_2}{\delta X_3} + \frac{\delta u_3}{\delta X_2}\right) = \frac{1}{2}\big((-8X_3 - X_2 + 5X_1 + 2) + (-2X_2 + 4X_1 - 3X_3 + 3)\big) = 4{,}5X_1 - 1{,}5X_2 - 5{,}5X_3 + 2{,}5$ [x10^{-4} m]
$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = (4X_1 + 3X_2 + 5X_3 - 2) + (-3X_1 + 4X_2 - X_3 + 6) + (2X_1 - 3X_2 + 6X_3 + 1) = 3X_1 + 4X_2 + 10X_3 + 5$ [x10^{-4} m]

2. Funkcje elementów tensora naprężeń:

Stałe materiałowe:
Moduł Younga: $E = 300{,}00\ GPa$
Współczynnik Poissona: $\nu = 0{,}30$

$$G = \frac{E}{2 \cdot (1 + \nu)} = \frac{3000{,}00}{2 \cdot (1 + 0{,}30)} = 115{,}385\ GPa$$
$$\lambda = \frac{E \cdot \nu}{(1 + \nu) \cdot (1 - 2\nu)} = \frac{300{,}00 \cdot 0{,}30}{(1 + 0{,}30) \cdot (1 - 2 \cdot 0{,}30)} 173{,}077\ GPa$$

$\sigma_{11} = 2G\varepsilon_{11} + \lambda\varepsilon_{kk} = 2 \cdot 11{,}5385 \cdot (4X_1 + 3X_2 + 5X_3 - 2) + 17{,}3077 \cdot (3X_1 + 4X_2 + 10X_3 + 5) = 144{,}231X_1 + 138{,}462X_2 + 288{,}462X_3 + 40{,}384$ [MPa]
$\sigma_{22} = 2G\varepsilon_{22} + \lambda\varepsilon_{kk} = 2 \cdot 11{,}5385 \cdot (-3X_1 + 4X_2 - X_3 + 6) + 17{,}3077 \cdot (3X_1 + 4X_2 + 10X_3 + 5) = 17{,}308X_1 + 161{,}539X_2 + 150{,}000X_3 + 225{,}000$ [MPa]
$\sigma_{33} = 2G\varepsilon_{33} + \lambda\varepsilon_{kk} = 2 \cdot 11{,}5385 \cdot (2X_1 - 3X_2 + 6X_3 + 1) + 17{,}3077 \cdot (3X_1 + 4X_2 + 10X_3 + 5) = 98{,}077X_1 + 311{,}539X_3 + 109{,}615$ [MPa]
$\sigma_{12} = 2G\varepsilon_{12} = 2 \cdot 11{,}5385 \cdot (1{,}5X_1 + 6{,}5X_2 + 4{,}5X_3 + 1{,}5) = 34{,}615X_1 + 150{,}000X_2 + 103{,}846X_3 + 34{,}615$ [MPa]

$\sigma_{13} = 2G\varepsilon_{13} = 2 \cdot 11{,}5385 \cdot (4{,}5X_1 + 4X_2 + 3{,}5) = 103{,}846X_1 + 92{,}308X_2 + 80{,}769$ [MPa]
$\sigma_{12} = 2G\varepsilon_{23} = 2 \cdot 11{,}5385 \cdot (4{,}5X_1 - 1{,}5X_2 - 5{,}5X_3 + 2{,}5\ ) = 103{,}846X_1 - 34{,}615X_2 - 126{,}923X_3 + 57{,}692$ [MPa]

3. Składowe sił masowych w punkcie A:

Współrzędne punktu A:
A=(2; 5; -1)

$$\frac{\delta\sigma_{11}}{\delta X_1} + \frac{\delta\sigma_{12}}{\delta X_2} + \frac{\delta\sigma_{13}}{\delta X_3} + P_1 = 0$$
$$144{,}231 + 34{,}615 + 103{,}846 + P_1 = 0 \Rightarrow P_1 = -282{,}692\ [MN/m^3]$$

$$\frac{\delta\sigma_{21}}{\delta X_1} + \frac{\delta\sigma_{22}}{\delta X_2} + \frac{\delta\sigma_{23}}{\delta X_3} + P_2 = 0$$
$$34{,}615 + 161{,}539 + (-126{,}923) + P_2 = 0 \Rightarrow P_2 = -69{,}231\ [MN/m^3]$$

$$\frac{\delta\sigma_{31}}{\delta X_1} + \frac{\delta\sigma_{32}}{\delta X_2} + \frac{\delta\sigma_{33}}{\delta X_3} + P_3 = 0$$
$$103{,}846 + (-34{,}615) + 311{,}539 + P_3 = 0 \Rightarrow P_3 = -380{,}770\ [MN/m^3]$$

4. Wartości elementów tensora naprężeń i odkształceń:

$X_1 = 2;\ \ X_2 = 5;\ X_3 = -1$

4.1. Naprężenia:

$$[\sigma_{ij}]_A = \begin{bmatrix} 732{,}694 & 749{,}999 & 750{,}001 \\ 749{,}999 & 848{,}079 & 219{,}232 \\ 750{,}001 & 219{,}232 & -5{,}770 \end{bmatrix}\ [MPa]$$

4.2. Odkształcenia:

$$[\varepsilon_{ij}]_A = \begin{bmatrix} 16{,}000 & 32{,}500 & 21{,}000 \\ 32{,}500 & 21{,}000 & 9{,}500 \\ 21{,}000 & 9{,}500 & -16{,}000 \end{bmatrix}\ [x10^{-4}]$$

5. Wartości i kierunki naprężeń głównych w punkcie A:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$
$$I_1 = 732{,}694 + 848{,}079 + (-5{,}770) = 1575{,}003$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix}$$
$$I_2 = \begin{vmatrix} 732{,}694 & 749{,}999 \\ 749{,}999 & 848{,}079 \end{vmatrix} + \begin{vmatrix} 848{,}079 & 219{,}232 \\ 219{,}232 & (-5{,}770) \end{vmatrix} + \begin{vmatrix} 732{,}694 & 750{,}001 \\ 750{,}001 & (-5{,}770) \end{vmatrix}$$
$$= 58883{,}894 + (-52956{,}086) + (-566729{,}144) = -560801{,}336$$

$$I_3 = |\sigma|$$
$$I_3 = \begin{vmatrix} 732{,}694 & 749{,}999 & 750{,}001 \\ 749{,}999 & 848{,}079 & 219{,}232 \\ 750{,}001 & 219{,}232 & -5{,}770 \end{vmatrix} = -265964699{,}500$$

$$\sigma^3 - 1575{,}003\sigma^2 + (-560801{,}336)\sigma - (-265964699{,}500) = 0$$
$$\Rightarrow \sigma_I = 1804{,}133\ [MPa]$$
$$\sigma_{II} = 286{,}115\ [MPa]$$
$$\sigma_{III} = -515{,}246\ [MPa]$$

Sprawdzenie:

$$\begin{cases}(\sigma_{11} - \sigma)\alpha_{1n} + \sigma_{12}\alpha_{2n} + \sigma_{13}\alpha_{3n} = 0\\ \sigma_{21}\alpha_{1n} + (\sigma_{22} - \sigma)\alpha_{2n} + \sigma_{23}\alpha_{3n} = 0\\ \sigma_{31}\alpha_{1n} + \sigma_{32}\alpha_{2n} + (\sigma_{33} - \sigma)\alpha_{3n} = 0\end{cases}$$

$$\alpha_{1n}^2 + \alpha_{2n}^2 + \alpha_{3n}^2 = 1$$

$$\sigma_I = 1804{,}133\ [MPa] \Rightarrow \begin{cases}(732{,}694 - 1804{,}133)\alpha_{11} + 749{,}999\alpha_{21} + 750{,}001\alpha_{31} = 0\\ 749{,}999\alpha_{11} + (848{,}079 - 1804{,}133)\alpha_{21} + 219{,}232\alpha_{31} = 0\\ 750{,}001\alpha_{11} + 219{,}232\alpha_{21} + (-5{,}770 - 1804{,}133)\alpha_{31} = 0\end{cases}$$

$$\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{31}^2 = 1 \Rightarrow \alpha_{31}^2 = 1 - \alpha_{11}^2 - \alpha_{21}^2$$

$$\begin{cases}-1071{,}439\alpha_{11} + 749{,}999\alpha_{21} + 750{,}001\alpha_{31} = 0\\ 749{,}999\alpha_{11} - 1175{,}286\alpha_{21} + 219{,}232\alpha_{31} = 0\\ \alpha_{31}^2 = 1 - \alpha_{11}^2 - \alpha_{21}^2\end{cases} \Rightarrow \begin{cases}\alpha_{11} = 0{,}35999\\ \alpha_{21} = 0{,}32567\\ \alpha_{31} = 0{,}87427\end{cases}$$

$$0{,}35999^2 + 0{,}32567^2 + 0{,}87427^2 = 1{,}03033 \approx 1$$

$$\sigma_{II} = 286{,}115\ \ [MPa] \Rightarrow \begin{cases}(732{,}694 - 286{,}115)\alpha_{12} + 749{,}999\alpha_{22} + 750{,}001\alpha_{32} = 0\\ 749{,}999\alpha_{12} + (848{,}079 - 286{,}115)\alpha_{22} + 219{,}232\alpha_{32} = 0\\ 750{,}001\alpha_{12} + 219{,}232\alpha_{22} + (-5{,}770 - 286{,}115)\alpha_{32} = 0\end{cases}$$

$$\alpha_{12}^2 + \alpha_{22}^2 + \alpha_{32}^2 = 1 \Rightarrow \alpha_{32}^2 = 1 - \alpha_{12}^2 - \alpha_{22}^2$$

$$\begin{cases}-303{,}422\alpha_{12} + 749{,}999\alpha_{22} + 750{,}001\alpha_{32} = 0\\ 749{,}999\alpha_{12} + 342{,}732\alpha_{22} + 219{,}232\alpha_{32} = 0\\ \alpha_{32}^2 = 1 - \alpha_{12}^2 - \alpha_{22}^2\end{cases} \Rightarrow \begin{cases}\alpha_{12} = 0{,}47509\\ \alpha_{22} = -0{,}85868\\ \alpha_{32} = 0{,}19220\end{cases}$$

$$0{,}47509^2 + (-0{,}85868)^2 + 0{,}19220^2 = 0{,}99998 \approx 1$$

$$\sigma_{III} = -515{,}246\ \ [MPa] \Rightarrow \begin{cases}(732{,}694 + 515{,}246)\alpha_{13} + 749{,}999\alpha_{23} + 750{,}001\alpha_{33} = 0\\ 749{,}999\alpha_{13} + (848{,}079 + 515{,}246)\alpha_{23} + 219{,}232\alpha_{33} = 0\\ 750{,}001\alpha_{13} + 219{,}232\alpha_{23} + (-5{,}770 + 515{,}246)\alpha_{33} = 0\end{cases}$$

$$\alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2 = 1 \Rightarrow \alpha_{33}^2 = 1 - \alpha_{13}^2 - \alpha_{23}^2$$

$$\begin{cases}1247{,}940\alpha_{13} + 749{,}999\alpha_{23} + 750{,}001\alpha_{33} = 0\\ 749{,}999\alpha_{13} + 1363{,}325\alpha_{23} + 219{,}232\alpha_{33} = 0\\ \alpha_{33}^2 = 1 - \alpha_{13}^2 - \alpha_{23}^2\end{cases} \Rightarrow \begin{cases}\alpha_{13} = -0{,}80215\\ \alpha_{23} = 0{,}27008\\ \alpha_{33} = 0{,}53256\end{cases}$$

$$(-0{,}80215)^2 + 0{,}27008^2 + 0{,}53256^2 = 1{,}00001 \approx 1$$

6. Wartości ekstremalne naprężeń stycznych w punkcie A:

$$\tau_1 = \frac{\sigma_{II} - \sigma_{III}}{2} = \frac{286{,}115 - (-515{,}246)}{2} = 400{,}680\ [MPa]$$
$$\sigma_1 = \frac{\sigma_{II} + \sigma_{III}}{2} = \frac{286{,}115 + (-515{,}246)}{2} = -114{,}565\ [MPa]$$

$$\tau_2 = \frac{\sigma_I - \sigma_{III}}{2} = \frac{1804{,}133 - (-515{,}246)}{2} = 1159{,}689\ [MPa]$$ ⇒**<u>naprężenia ekstremalne</u>**
$$\sigma_2 = \frac{\sigma_I + \sigma_{III}}{2} = \frac{1804{,}133 + (-515{,}246)}{2} = 644{,}443\ [MPa]$$

$$\tau_3 = \frac{\sigma_I - \sigma_{II}}{2} = \frac{1804{,}133 - 286{,}115}{2} = 759{,}009\ [MPa]$$
$$\sigma_3 = \frac{\sigma_I + \sigma_{II}}{2} = \frac{1804{,}133 + 286{,}115}{2} = 1045{,}124\ [MPa]$$

7. Wartości składowych wektora naprężeń w punkcie A na płaszczyźnie o wektorze normalnym n:

Współrzędne wektora n:

$$n_1 = \frac{1}{6} = 0{,}167$$
$$n_2 = \frac{1}{8} = 0{,}125 < \sqrt{1 - n_1^2} = 0{,}986$$
$$n_3 = \sqrt{1 - n_1^2 - n_2^2} = 0{,}978$$
$$|\vec{n}| = 1$$

$$f_i^{(n)} = \sigma_{ij} \cdot \alpha_{nj}$$

$$f_1^{(n)} = \sigma_{11}\alpha_{n1} + \sigma_{12}\alpha_{12} + \sigma_{13}\alpha_{n3}$$
$$f_1^{(n)} = 732{,}694 \cdot 0{,}167 + 749{,}999 \cdot 0{,}125 + 750{,}001 \cdot 0{,}978 = 949{,}611$$

$$f_2^{(n)} = \sigma_{21}\alpha_{n1} + \sigma_{22}\alpha_{12} + \sigma_{23}\alpha_{n3}$$
$$f_2^{(n)} = 749{,}999 \cdot 0{,}167 + 848{,}079 \cdot 0{,}125 + 219{,}232 \cdot 0{,}978 = 445{,}667$$

$$f_3^{(n)} = \sigma_{31}\alpha_{n1} + \sigma_{32}\alpha_{12} + \sigma_{33}\alpha_{n3}$$
$$f_3^{(n)} = 750{,}001 \cdot 0{,}167 + 219{,}232 \cdot 0{,}125 + (-5{,}770) \cdot 0{,}978 = 147{,}011$$

$$\sigma_n = n_i \cdot f_i^{(n)} = 0{,}167 \cdot 949{,}611 + 0{,}125 \cdot 445{,}667 + 0{,}978 \cdot 147{,}011 = 358{,}070\ [MPa]$$

$$\tau_n = \sqrt{\left|\vec{f}^{(n)}\right|^2 - \sigma_n^2}$$
$$\left|\vec{f}^{(n)}\right|^2 = f_i^{(n)} \cdot f_i^{(n)}$$
$$\tau_n = \sqrt{949{,}611^2 + 445{,}667^2 + 147{,}011^2 - 358{,}070^2} = 996{,}884\ [MPa]$$

$$\vec{n} \cdot \vec{f}^{(n)} = |\vec{n}| \cdot \left|\vec{f}^{(n)}\right| \cdot cos\varphi = \sigma_n \Rightarrow cos\varphi = \frac{\sigma_n}{\left|\vec{f}^{(n)}\right|}$$
$$cos\varphi = \frac{358{,}070}{\sqrt{949{,}611^2 + 445{,}667^2 + 147{,}011^2}} = 0{,}338 \Rightarrow \varphi = 70^{\circ}14''$$